Strategy complexity of finite-horizon Markov decision processes and simple stochastic games*

Krishnendu Chatterjee[†]

Rasmus Ibsen-Jensen[‡]

Abstract

Markov decision processes (MDPs) and simple stochastic games (SSGs) provide a rich mathematical framework to study many important problems related to probabilistic systems. MDPs and SSGs with finite-horizon objectives, where the goal is to maximize the probability to reach a target state in a given finite time, is a classical and well-studied problem. In this work we consider the strategy complexity of finite-horizon MDPs and SSGs. We show that for all $\epsilon > 0$, the natural class of counter-based strategies require at most $\log \log(\frac{1}{\epsilon}) + n + 1$ memory states, and memory of size $\Omega(\log \log(\frac{1}{\epsilon}) + n)$ is required, for ϵ -optimality, where n is the number of states of the MDP (resp. SSG). Thus our bounds are asymptotically optimal. We then study the periodic property of optimal strategies, and show a sub-exponential lower bound on the period for optimal strategies.

1 Introduction

Markov decision process and simple stochastic games. The class of Markov decision processes (MDPs) is a classical model for probabilistic systems that exhibit both stochastic and and deterministic behavior [4]. MDPs have been widely used to model and solve control problems for stochastic systems [3]: there, non-determinism represents the freedom of the controller to choose a control action, while the probabilistic component of the behavior describes the system response to control actions. Simple stochastic games (SSGs) enrich MDPs by allowing two types of non-determinism (angelic and demonic non-determinism) along with stochastic behavior [1]. MDPs and SSGs provide a rich mathematical framework to study many important problems related to probabilistic systems.

Finite-horizon objective. One classical problem widely studied for MDPs and SSGs is the *finite-horizon objective*. In a finite-horizon objective, a finite time horizon T is given and the goal of the player is to maximize the payoff within the time horizon T in MDPs (in SSGs against all strategies of the opponent). The complexity of MDPs and SSGs with finite-horizon objectives have been well

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[†]IST Austria. Email: krish.chat@ist.ac.at.

[‡]Department of Computer Science, Aarhus University, Denmark. E-mail: rij@cs.au.dk.

studied, with book chapters dedicated to them [3, 7]. The complexity results basically show that iterating the Bellman equation for T steps yield the desired result [3, 7]. While the computational complexity have been well-studied, perhaps surprisingly the strategy complexity has not received great attention. In this work we consider several problems related to the strategy complexity of MDPs and SSGs with finite-horizon objectives, where the objective is to reach a target state within a finite time horizon T.

Our contribution. In this work we consider the memory requirement for ϵ -optimal strategies, for $\epsilon>0$, and a periodic property of optimal strategies in finite-horizon MDPs and SSGs. A strategy is an ϵ -optimal strategy, for $\epsilon>0$, if the strategy ensures within ϵ of the optimal value against all strategies of the opponent. For finite-horizon objectives, the natural class of strategies are counterbased strategies, which has a counter to count the number of time steps. Our first contribution is to establish asymptotically optimal memory bounds for ϵ -optimal counter-based strategies, for $\epsilon>0$, in finite-horizon MDPs and SSGs. We show that ϵ -optimal counter-based strategies require at most memory of size $\log\log(\frac{1}{\epsilon})+n+1$ and memory of size $\Omega(\log\log(\frac{1}{\epsilon})+n)$ is required, where n is the size of the state space. Thus our bounds are asymptotically optimal. The upper bound holds for SSGs and the lower bound is for MDPs. We then consider the periodic (or regularity) property of optimal strategies. The period of a strategy is the number P such that the strategy repeats within every P steps (i.e., it is periodic with time step P). We show a sub-exponential lower bound on the period of optimal strategies for MDPs with finite-horizon objectives, by presenting a family of MDPs with n states where all optimal strategies are periodic and the period is $2^{\Omega(\sqrt{n \cdot \log(n)})}$.

Organization of the paper. The paper is organized as follows: In Section 2 we present all the relevant definitions related to stochastic games and strategies. In Section 3 we show that $\Theta(n + \log \log \epsilon^{-1})$ number of bits are necessary and sufficient for ϵ -optimal counter-based strategies, for all $\epsilon > 0$, in both finite-horizon MDPs and SSGs. In Section 4 we show that there are finite-horizon MDPs where all optimal strategies are periodic and have a period of $2^{\Omega(\sqrt{n \log n})}$.

2 Definitions

The class of *infinite-horizon simple stochastic games* (SSGs) consists of two player, zero-sum, turn-based games, played on a (multi-)graph. The class was first defined by Condon [1]. Below we define SSGs, the finite-horizon version, and the important sub-class of MDPs.

SSGs, finite-horizon SSGs, and MDPs. An SSG $G = (S_1, S_2, S_R, \bot, (A_s)_{s \in S_1 \cup S_2 \cup S_R}, s_0)$ consists of a terminal state \bot and three sets of disjoint non-terminal states, S_1 (max state), S_2 (min states), S_R (coin toss states). We will use S to denote the union, i.e., $S = S_1 \cup S_2 \cup S_R$. For each state $s \in S$, let A_s be a (multi-)set of outgoing arcs of s. We will use $A = \bigcup_s A_s$ to denote the (multi-)set of all arcs. Each state $s \in S$ has two outgoing arcs. If a is a arc, then $d(a) \in S \cup \{\bot\}$ is the destination of a. There is also a designated start state $s_0 \in S$. The class of finite-horizon simple stochastic games (FSSGs) also consists of two player, zero-sum, turn-based games, played on a (multi-)graph. An FSSG (G,T) consists of an SSG G and a finite time limit (or horizon) $T \ge 0$. Let G be an SSG and $T \ge 0$, then we will write the FSSG (G,T) as G^T . Given an SSG G (resp. FSSG G^T), for a state s, we denote by G_s (resp. G_s^T) the same game as G (resp. G_s^T), except that s is the start state. The class of infinite (resp. finite) horizon Markov decision processes (MDPs and FMDPs respectively) is the subclass of SSGs (resp. FSSGs) where $S_2 = \emptyset$.

Plays and objectives of the players. An SSG G is played as follows. A pebble is moved on to s_0 .

For $i \in \{1, 2\}$, whenever the pebble is moved on to a state s in S_i , then Player i chooses some arc $a \in A_s$ and moves the pebble to d(a). Whenever the pebble is moved on to a state s in S_R , then an $a \in A_s$ is chosen uniformly at random and the pebble moves to d(a). If the pebble is moved on to \bot , then the game is over. For all $T \ge 0$ the FSSG G^T is played like G, except that the pebble can be moved at most T+1 times. The *objective* of both SSGs and FSSGs is for Player 1 to maximize the probability that the pebble is moved on to \bot (eventually in SSGs and with in T+1 time steps in FSSGs). The objective of Player 2 is to minimize this probability.

Strategies. Let S^* be the set of finite sequences of states. For all T, let $S^{\leq T} \subset S^*$ be the set of sequences of states, which have length at most T. A strategy σ_i for Player i in an SSG is a map from $S^* \times S_i$ into A, such that for all $w \in S^*$ and $s \in S$ we have $\sigma_i(w \cdot s) \in A_s$. Similarly, a strategy σ_i for Player i in an FSSG G^T is a map from $S^{\leq T} \times S_i$ into A, such that for all $w \in S^{\leq T}$ and $s \in S$ we have $\sigma_i(w \cdot s) \in A_s$. In all cases we denote by Π_i the set of all strategies for Player i. If $S_i = \emptyset$, we will let \emptyset denote the corresponding strategy set. Below we define some special classes of strategies.

Memory-based, counter-based and Markov strategies. Let $M = \{0, 1\}^*$ be the set of possible memories. A memory-based strategy σ_i for Player i consists of a pair (σ_u, σ_a) , where

- σ_u , the memory-update function, is a map from $M \times S$ into M
- σ_a , the next-action function, is a map from $M \times S_i$ into A, such that for all $m \in M$ and $s \in S_i$ we have $\sigma_a(m,s) \in A_s$.

A counter-based strategy is a special case of memory-based strategies, where for all $m \in M$ and $s, s' \in S$ we have $\sigma_u(m, s) = \sigma_u(m, s')$. That is the memory can only contain a counter of some type. We will therefore write $\sigma_u(m, s)$ as $\sigma_u(m)$ for all m, s and any counter-based strategy σ . A Markov strategy σ_i for Player i is a special case of strategies where

$$\forall p, p' \in S^{\leq T} : |p| = |p'| \land p_{|p|} = p'_{|p'|} \in S_i \Rightarrow \sigma(p', p'_{|p'|}) = \sigma(p, p_{|p|}).$$

That is, a Markov strategy only depends on the length of the history and the current state. Let Π'_i be the set of all Markov strategies for Player i.

Following a strategy. For a strategy, σ_i , for Player i we will say that Player i follows σ_i if for all n given the sequence of states $(p_i)_{i \leq n}$ the pebble has been on until move n and that $p_n \in S_i$, then Player i chooses $\sigma((p_i)_{i \leq n}, p_n)$. For a memory-based strategy for Player i σ_i , we will say that Player i follows σ_i if for all n given the sequence of states $(p_i)_{i \leq n}$ the pebble has been on until move n, that $p_n \in S_i$ and that $m^i = \sigma_u(m^{i-1}, p_i)$ and that $m^0 = \emptyset$, then Player i chooses $\sigma_a(m^n, p_n)$.

Space required by a memory-based strategy. The space usages of a memory-based strategy is the logarithm of the number of distinct states generated by the strategy at any point, if the player follows that strategy. A memory-based strategy is memoryless if there is only one memory used by the strategy. For any FSSG G^T with n states it is clear that the set of strategies is a subset of memory-based strategies that uses memory at most $T \log n$, since for any strategy σ we can construct a memory-based strategy σ' by using the memory for the sequence of states and then choose the same action as σ would with that sequence of states. Hence we will also talk about ϵ -optimal memory-based strategies. Also note that for any FSSG G^T it is clear that the set of Markov strategies is a subset of the set of counter-based strategies that uses space at most $\log T$.

Period of a counter-based strategy. We will distinguish between two kinds of memories for a counter-based strategy σ . One kind is only used once (the initial phase) and the other kind is used arbitrarily

many times (the periodic phase). Let $m^0 = \emptyset$ and $m^i = \sigma_u(m^{i-1})$. Then if $m^i = m^j$ for some i < j, we also have that $m^{i+c} = m^{j+c}$ and $m^i = m^{i+c(j-i)}$. Hence if a memory is used twice, it will be reused again. We will let the number of memories that are only used once be N and the number of memories used more than once be p, which we will call the period. The number N is mainly important for ϵ -optimal strategies and period is mainly important for optimal strategies.

Probability measure and values. A pair of strategies (σ_1, σ_2) , one for each player (in either an SSG or an FSSG), defines a probability that the pebble is eventually moved to \bot . Let the probability be denoted as P^{σ_1,σ_2} . For all SSGs G (resp. FSSGs G^T) it follows from the results of Everett [2] that

$$\sup_{\sigma_1 \in \Pi_1'} \inf_{\sigma_2 \in \Pi_2} P^{\sigma_1, \sigma_2} = \inf_{\sigma_2 \in \Pi_2'} \sup_{\sigma_1 \in \Pi_1} P^{\sigma_1, \sigma_2}.$$

We will call this common value as the value of G (resp. G^T) and denote it val(G) (resp. val (G^T)). ϵ -optimal and optimal strategies. For all $\epsilon \geq 0$, we will say that a strategy σ_1 is ϵ -optimal for Player 1 if

$$\inf_{\sigma_2 \in \Pi_2} P^{\sigma_1, \sigma_2} + \epsilon \ge \sup_{\sigma_1' \in \Pi_1'} \inf_{\sigma_2 \in \Pi_2} P^{\sigma_1', \sigma_2}.$$

Similarly, a strategy σ_2 is ϵ -optimal for Player 2 if

$$\sup_{\sigma_1 \in \Pi_1} P^{\sigma_1,\sigma_2} - \epsilon \leq \inf_{\sigma_2' \in \Pi_2'} \sup_{\sigma_1 \in \Pi_1} P^{\sigma_1,\sigma_2'}.$$

A strategy σ is optimal for Player i if it is 0-optimal. Condon [1] showed that there exist optimal memoryless strategies for any SSG G that are also optimal for G_s for all $s \in S$. This also implies that there are optimal Markov strategies for FSSGs that are also optimal for G_s for all $s \in S$.

3 Bounds on ϵ -optimal counter-based strategies

We will first show an upper bound on size of the memory used by a counter-based strategy for playing ϵ -optimal in time limited games. The upper bound on memory size is by application of a result from Ibsen-Jensen and Miltersen [5]. The idea of the proof is that if we play an optimal strategy of G in G^T for sufficiently high T, then the value we get approaches the value of G.

Theorem 1 (Upper bound) For all FSSGs G^T with n states and $\epsilon > 0$, there is an ϵ -optimal counter-based strategy for both players such that memory size is at most $\log \log \epsilon^{-1} + n + 1$

Proof Since there is an optimal Markov strategy, there is a counter-based strategy, which uses memory at most $\log T$. As shown by Ibsen-Jensen and Miltersen [5] for any game G^T , if the horizon is greater than $2\log \epsilon^{-1}2^n$, the value of G^T approximates the value of G with in ϵ . It is clear that the value of all states are the same in an infinite-horizon game if either player is forced to play an optimal strategy. Hence, if $T \geq 2\log \epsilon^{-1}2^n$ and either player plays an optimal strategy of G in G^T , then the value of all states are within ϵ of the value of the game. But there are optimal memoryless strategies in G as shown by Condon [1]. Therefore we have that in the worst case $T < 2\log \epsilon^{-1}2^n$. Since $\log T$ is an upper bound, $\log \log \epsilon^{-1} + n + 1$ is also an upper bound and hence the result. \square

We will now lower bound the size of the memory needed for a counter-based strategy to be ϵ -optimal. Our lower bound will be divided into two parts. The first part will show that $\log \log \epsilon^{-1}$

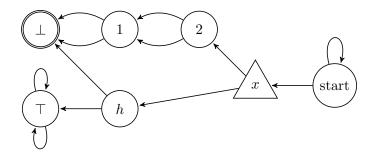


Figure 1: An MDP G, such that for all $\epsilon > 0$ there is a T, such that all ϵ -optimal memory-based strategies for G^T require memory size of at least $\Omega(\log \log \epsilon^{-1})$. Circle vertices are the coin toss states. The triangle vertex is the max state. The vertex \bot is the terminal state.

is a lower bound on the memory required even for some MDPs with constantly many states. The second part will show that even for fixed ϵ , an ϵ -optimal counter-based strategy will need to use a memory of size O(n). Both lower bounds will show explicit MDPs with the required properties. See Figure 1 and Figure 2 respectively.

MDP for the lower bound of $\log \log \epsilon^{-1}$. Our first lower bound shows that in the MDP M (Figure 1) all ϵ -optimal memory-based strategies require at least $\log \epsilon^{-1}$ distinct memory states, i.e., the size of memory is at least $\log \log \epsilon^{-1}$. The MDP M is defined as follows. There is one state x in S_1 , the rest are in S_R .

- The state $\top \in S_R$ has $A_{\top} = \{(\top, \top), (\top, \top)\}.$
- The state $h \in S_R$ has $A_h = \{(h, \top), (h, \bot)\}.$
- The state $1 \in S_R$ has $A_1 = \{(1, \perp), (1, \perp)\}.$
- The state $2 \in S_R$ has $A_2 = \{(2,1), (2,1)\}.$
- The state $x \in S_1$ has $A_m = \{(x, 2), (x, h)\}.$
- The state start $\in S_R$ has $A_{\text{start}} = \{(\text{start}, \text{start}), (\text{start}, x)\}.$

Lemma 2 All ϵ -optimal memory-based strategies in M^T , for $T = \log \epsilon^{-1} - 1$, require at least $\log \epsilon^{-1} - 2$ distinct states of memory, i.e., the size of memory is at least $\log \log \epsilon^{-1}$.

Proof We will first show the proof for counter-based strategies. At the end we will then extend it to memory-based strategies.

It is clear that $\operatorname{val}(M_x^2) = \frac{1}{2}$ and for all T > 2 we have $\operatorname{val}(M_x^T) = 1$. If Player 1 chooses (x,h) in M_x^2 , then he gains $\frac{1}{2}$, otherwise, if he chooses (x,2), then he gains 0. Also for all T > 2, if Player 1 chooses (x,2) in M_x^T , then he gains 1, otherwise, if he chooses (x,h), then he gains $\frac{1}{2}$.

In M_{start} we end up at x after precisely $k \geq 2$ moves of the pebble with probability 2^{-k+1} . Therefore, by the preceding any optimal memory-based strategy σ must be able to find out if T minus the length of the history is greater than 2 from the memory.

Let $\epsilon > 0$ be given. For simplicity we will assume that $\epsilon = 2^{-k}$ for some k > 0. Let $c = \log \epsilon^{-1}$. Assume now that there is a counter-based strategy $\sigma = (\sigma_u, \sigma_a)$ that uses c - 3 states of memory

in $M_{\rm start}^{c-1}$. The pebble ends up at m after c-3 moves with probability $2^{-(c-3)+1}=4\epsilon$. Let the sequences of memories until then be $m^0, m^1, \ldots, m^{c-3}$. Since σ was ϵ -optimal we must have that $\sigma(m^{c-3}, x) = (x, h)$. On the other hand for all i < c-3 we must also have that $\sigma(m^i, x) = (x, 2)$. Therefore m^{c-3} differs from m^i for i < c-3. Now assume that $m^i = m^j$ for i < j and i, j < c-3. But then $\sigma_u(m^i) = \sigma_u(m^j)$ and hence $m^{i+1} = m^{j+1}$ and then by repeating this argument we have that $m^k = m^{c-3}$ for k < c-3. Therefore m^i differs from m^j for $i \neq j$ and $i, j \leq c-3$ and hence we need at least c-2 different memory states.

For general memory-based strategies the proof remains the same. This is because we can note that if the pebble ends up at x after c-3 moves, we have that $m^0 = \emptyset$ and $m^i = \sigma_u(m^{i-1}, \text{start})$ for $1 \le i \le c-3$ and hence they must all differ by the same argument as before.

For our second lower bound we will use an infinite family of MDPs

$$H = \{H(1), H(2), \dots, H(i), \dots\},\$$

such that H(i) contains 2i + 4 states, one of which is a max state, and all ϵ -optimal counter-based strategies require space at least i - 4, for some fixed ϵ .

Family of MDPs for the lower bound of n. The MDP H(i) is defined as follows. There is one state x in S_1 , the rest are in S_R .

- The state $\top \in S_R$ has $A_\top = \{(\top, \top), (\top, \top)\}.$
- The state $h \in S_R$ has $A_h = \{(h, \top), (h, \bot)\}.$
- The state $1 \in S_R$ has $A_1 = \{(1, \perp), (1, i)\}.$
- For $j \in \{2, ..., i\}$, the state $j \in S_R$ has $A_j = \{(j, i), (j, j 1)\}$.
- The state $x \in S_1$ has $A_m = \{(x, i), (x, h)\}.$
- The state $1^* \in S_R$ has $A_{1^*} = \{(1^*, i^*), (1^*, x)\}.$
- For $j \in \{2, ..., i\}$, the state $j^* \in S_R$ has $A_{j^*} = \{(j^*, i^*), (j^*, (j-1)^*)\}$.

There is a illustration of H(4) in Figure 2.

Let i be some number. It is clear that $\operatorname{val}(H(i)_x^2) = \frac{1}{2}$. It is also easy to see that $\operatorname{val}(H(i)_i) = 1$, but that the time to reach \bot from i is quite long. Hence, one can deduce that there must be a k (k depends on i) such that for all $k' \ge k$ it is an optimal strategy in $H(i)_x^{k'}$ to choose (x,i) and for all $1 \le k'' < k$ it is an optimal strategy in $H(i)_x^{k''}$ to choose (x,k). In case there are multiple such numbers, let k be the smallest. The number k-1 is then the smallest number of moves of the pebble to reach \bot from i, such that that occurs with probability $1 \le k$ (to simplify the proofs we will assume equality).

Let p^t be the probability for the pebble to reach x from i^* in t or less moves (note that this is also the probability to reach \bot in t moves or less from i). It is clear that p^t is equal to the probability that a sequence of t fair coin tosses contains i consecutive tails. This is known to be exactly $1 - F_{t+2}^{(i)}/2^t$, where $F_{t+2}^{(i)}$ is the (t+2)'nd Fibonacci i-step number, i.e. the number given by the linear homogeneous recurrence $F_c^{(i)} = \sum_{j=1}^i F_{c-j}^{(i)}$ and the boundary conditions $F_c^{(i)} = 0$, for $c \le 0$, $F_1^{(i)} = F_2^{(i)} = 1$ (this fact is also mentioned in Ibsen-Jensen and Miltersen [5]).

The next lemmas will prove various properties of p^t , $F_a^{(i)}$ and k. We will first show two technical lemmas that will be used in many of the remaining lemmas. Next, we will show that k is exponential in i and show various bounds on p^t . We will use all that to show that the number of states in the game is a lower bound on the memory requirement for ϵ -optimal counter-based strategies.

Lemma 3 Let i and $a \ge i + 3$ be given. Then

$$F_a^{(i)} \le (2 - 2^{-i-1}) F_{a-1}^{(i)}$$

Let $b \geq 3$ be given. Then

$$F_b^{(i)} \le 2F_{b-1}^{(i)}$$

Proof We can see that

$$F_b^{(i)} = \sum_{j=1}^{i} F_{b-j}^{(i)} = 2F_{b-1}^{(i)} - F_{b-i}^{(i)}$$

for $b \geq 3$. Hence we have that $F_b^{(i)} \leq 2F_{b-1}^{(i)}$. We therefore have that $F_{a-i}^{(i)} \geq 2^{-i-1}F_{a-1}^{(i)}$ and we can deduce that

$$F_a^{(i)} \le 2F_{a-1}^{(i)} - 2^{-i-1}F_{a-1}^{(i)}.$$

The desired result follows.

Now for the proof that k is exponential in i.

Lemma 4 For all i, we have that $k \geq 2^{i-2} + i$.

Proof We will first show that $p^a < p^{a-1} + 2^{-i}$. We can divide the event that there are i consecutive tails into two possibilities out of t fair coin tosses. Either the first i coin tosses were tails or there are i consecutive tails in the last t-1 coin tosses (or both). The first case happens with probability 2^{-i} and the last with probability p^{a-1} . We can then apply union bounds and get that $p^a \leq p^{a-1} + 2^{-i}$. Clearly we have that $p^{i-1}=0$ and that p^a is increasing in a. But we also have that

$$\begin{split} p^k & \leq 2^{i-2} 2^{-i} + p^{k-2^{i-2}} \Rightarrow \\ \frac{1}{2} & \leq \frac{1}{4} + p^{k-2^{i-2}} \Rightarrow \\ \frac{1}{4} & \leq p^{k-2^{i-2}}, \end{split}$$

which means that $k > 2^{i-2} + i - 1$.

Lemma 5 Let i be given. The number k is such that

$$e^{\frac{-1}{8}} \ge (1 - 2^{-i-2})^k \ge \frac{1}{4}$$

and such that

$$e^{\frac{-1}{8}} \ge (1 - 2^{-i-2})^{k-i} \ge \frac{1}{2}$$

Proof We have that $1 - F_{k+2}^{(i)}/2^k = \frac{1}{2}$, which we can then use to show that

$$1 - F_{k+2}^{(i)}/2^k = \frac{1}{2} \Rightarrow$$

$$F_{k+2}^{(i)}/2^k = \frac{1}{2} \Rightarrow$$

$$\frac{(2 - 2^{-i-1})^{k-i}2^i F_2^{(i)}}{2^k} \ge \frac{1}{2} \Rightarrow$$

$$(1 - 2^{-i-2})^{k-i} \ge \frac{1}{2}$$

where we used Lemma 3 for the second implication. We used that $F_2^{(i)} = 1$ for the third implication. Since $k \ge 2^{i-2} + i > 2i$ by Lemma 4, we also have that $(1 - 2^{-i-2})^k \ge \frac{1}{4}$.

But we can also use Lemma 4 more directly. Notice that since $i \ge 12$ we have that $2^{i+2} \ge 72$. We have that,

$$(1 - 2^{-i-2})^{k-i} \le (1 - 2^{-i-2})^{2^{i-2}} = ((1 - 2^{-i-2})^{2^{i+2}})^{\frac{1}{8}} \le e^{\frac{-1}{8}},$$

where we used that $\lim_{x\to\infty} (1-x^{-1})^x = e^{-1}$ and that $(1-x^{-1})^x$ is increasing in x for $x\geq 1$. We also have that $e^{\frac{-1}{8}}\geq (1-2^{-i-2})^k$, by the same argument.

Lemma 6 For all i and t, we have

$$p^{2t-2i} \le 2p^t$$

Proof Let t' = t - i. Hence, we need to show that $p^{2t'} \leq 2p^{t'+i}$. The proof comes from the fact that to have i consecutive tails out of 2t' fair coin tosses, the i consecutive tails must either start in the first half or end in the second half (or both). But to start in the first half means that it must end in the first t' + i elements. Therefore we can overestimate that probability with $p^{t'+i}$. Similar with the second half. We can then add them together by union bound and the result follows. \square

Lemma 7 Let $i \ge 12$ and $\frac{1}{10} < d < 1$ be given. Then $p^{dk} \le 1 - \frac{e^{\frac{1-d}{8}}}{2} < \frac{1}{2}$.

Proof Since $d > \frac{1}{10}$, we have that dk > i, by Lemma 4 and because $i \ge 12$. We will show that $F_{dk+2}^{(i)}/2^{dk} \ge \frac{e^{\frac{1-d}{8}}}{2}$. We have that

$$F_{dk+2}^{(i)}/2^{dk} \ge \frac{F_{k+2}^{(i)}}{(2-2^{-i-1})^{(1-d)k}2^{dk}}$$

$$= \frac{F_{k+2}^{(i)}}{(1-2^{-i-2})^{(1-d)k}2^k}$$

$$= \frac{1}{2 \cdot (1-2^{-i-2})^{(1-d)k}}$$

$$= \frac{1}{2 \cdot ((1-2^{-i-2})^k)^{1-d}}$$

$$\ge \frac{1}{2 \cdot (e^{-\frac{1}{8}})^{1-d}}$$

$$= \frac{e^{\frac{1}{8}(1-d)}}{2}$$

where we used Lemma 3 for the first inequality, Lemma 5 for the second and that $\lim_{x\to\infty} (1-x^{-1})^x = e^{-1}$ and that $(1-x^{-1})^x$ is increasing in x for $x \ge 1$ for the third.

Lemma 8 Let $i \ge 12$ and 0 < d be given. Then $p^{(1+d)k} \ge 1 - (e^{\frac{-d}{8}})\frac{1}{2} > \frac{1}{2}$.

Proof We will show that $F_{(1+d)k+2}^{(i)}/2^{(1+d)k} \leq (e^{\frac{-d}{8}})\frac{1}{2}$. We have that

$$F_{(1+d)k+2}^{(i)}/2^{dk} \le \frac{(2-2^{-i-1})^{dk} F_{k+2}^{(i)}}{2^{(1+d)k}}$$

$$= (1-2^{-i-2})^{dk} \frac{1}{2}$$

$$= ((1-2^{-i-2})^k)^d \frac{1}{2}$$

$$\le (e^{\frac{-1}{8}})^d \frac{1}{2}$$

$$= (e^{\frac{-d}{8}}) \frac{1}{2}$$

where we used Lemma 3 for the first inequality and Lemma 5 for the second.

Lemma 9 There is an ϵ such that for all $i \geq 12$, there is a time-bound T such that all ϵ -optimal counter-based strategies for $H(i)^T$ require memory size at least i-5.

The proof basically goes as follows: The pebble starts at i^* with 2k+1 moves remaining. First we show that there is a super-constant probability for the pebble to reach x using somewhere between $\frac{k}{5}$ and $\frac{4k}{5}$ moves. In that case there is at least $\frac{6k}{5}+1$ moves left. We then show that there is some number $p>\frac{1}{2}$ independent of i such that the probability to reach \perp from i in $\frac{6k}{5}$ is more than p. Secondly we show that there is a super-constant probability for the pebble to reach x using somewhere between $\frac{6k}{5}$ and $\frac{9k}{5}$ moves. In that case there is at most $\frac{4k}{5}+1$ moves left. We then show that there is some number $q<\frac{1}{2}$ independent of i such that the probability to reach \perp from i in $\frac{4k}{5}$ is less than q. We can then pick ϵ such that any ϵ -optimal strategy must distinguish between plays that used between $\frac{k}{5}$ and $\frac{4k}{5}$ moves to reach x from i^* and plays that used between $\frac{6k}{5}$ and $\frac{9k}{5}$ moves to reach x from x from

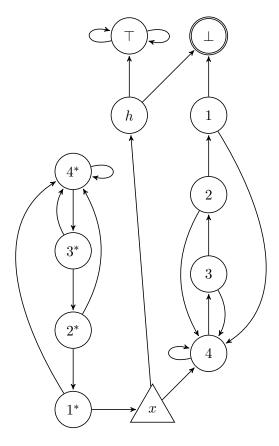


Figure 2: The MDP H(4). It is the fourth member of a family that will show that there exist FSSGs where, for a fixed ϵ , all ϵ -optimal counter-based strategies require memory size to be at least $\Omega(i)$. Circle vertices are the coin toss states. The triangle vertex is the max state. The vertex \bot is the terminal state.

Proof The probability for the pebble to reach x using somewhere between $\frac{k}{5}$ and $\frac{4k}{5}$ moves is

$$p^{\frac{4k}{5}} - p^{\frac{k}{5}} = 1 - F_{\frac{4k}{5}+2}^{(i)} / 2^{\frac{4k}{5}} - (1 - F_{\frac{k}{5}+2}^{(i)} / 2^{\frac{k}{5}})$$

$$= \frac{2^{\frac{3k}{5}} F_{\frac{k}{5}+2}^{(i)} - F_{\frac{4k}{5}+2}^{(i)}}{2^{\frac{4k}{5}}}$$

$$\geq \frac{2^{\frac{3k}{5}} F_{\frac{k}{5}+2}^{(i)} - (2 - 2^{-i-1})^{\frac{3k}{5}} F_{\frac{k}{5}+2}^{(i)}}{2^{\frac{4k}{5}}}$$

$$= \frac{(2^{\frac{3k}{5}} - (2 - 2^{-i-1})^{\frac{3k}{5}}) F_{\frac{k}{5}+2}^{(i)}}{2^{\frac{4k}{5}}}$$

$$= \frac{(1 - (1 - 2^{-i-2})^{\frac{3k}{5}}) F_{\frac{k}{5}+2}^{(i)}}{2^{\frac{k}{5}}}$$

$$= (1 - (1 - 2^{-i-2})^k) (1 - p^{\frac{k}{5}})$$

$$\geq (1 - e^{\frac{-1}{8}}) \frac{e^{\frac{1}{10}}}{2}$$

where we used Lemma 3 for the first inequality and Lemma 5 and Lemma 7 for the second.

In this case we have at least $\frac{6k}{5} + 1$ moves left. Therefore if the player chooses to move to i, there are at least $\frac{6k}{5}$ moves left. In that case, by Lemma 8, the pebble will reach \perp with probability at least $1 - (e^{\frac{-3}{40}})\frac{1}{2} > \frac{1}{2}$. In both cases we see that the probability is strictly separated from $\frac{1}{2}$.

The probability for the pebble to reach x using somewhere between $\frac{6k}{5}$ and $\frac{9k}{5}$ moves can be calculated similar to between $\frac{k}{5}$ and $\frac{4k}{5}$ moves. We end up with

$$p^{\frac{9k}{5}} - p^{\frac{6k}{5}} \ge (1 - e^{\frac{-1}{8}})(1 - p^{\frac{6k}{5}}).$$

Hence, we need a upper bound on $p^{\frac{6k}{5}}$, which is smaller than 1 and does not depend on k or i. We can get that by noting that $\frac{6k}{5} \leq \frac{8k}{5} - 2i$, because of Lemma 4 and that $i \geq 12$. Hence we can apply Lemma 6 followed by Lemma 7 and get that $p^{\frac{6k}{5}} \leq 2p^{\frac{4k}{5}} \leq 2(1 - \frac{e^{\frac{1}{40}}}{2}) < 1$.

In this case we have at most $\frac{4k}{5} + 1$ moves left. Therefore if the player chooses to move to i, there are at most $\frac{4k}{5}$ moves left. In that case, by Lemma 7, the pebble will reach \perp with probability at most $1 - \frac{e^{\frac{1}{40}}}{2} < \frac{1}{2}$.

Let σ be some ϵ -optimal counter-based strategy and assume that σ uses less than $\frac{k}{5}-1$ states. We will show that if ϵ is some sufficiently low constant, we get a contradiction and hence all ϵ -optimal counter-based strategies uses at least $\frac{k}{5}$ states. Our result than follows from Lemma 4.

Let $m^0 = \emptyset$ and $m^i = \sigma_u(m^{i-1})$. Since σ uses less than $\frac{k}{5}$ states, then $m^a = m^b$ for some $a < b < \frac{k}{5}$. Hence also $m^{a+c} = m^{b+c}$ for all $c \ge 0$, by definition. But then $m^{a+c} = m^{a+c+(b-a)d}$ for all c and d greater than 0. Hence, we can make a one to one map between memory m^a for $a \in A = \{\frac{k}{5}, \dots, \frac{4k}{5}\}$ and some memory m^b for $b \in B = \{\frac{6k}{5}, \dots, \frac{9k}{5}\}$, such that $m^a = m^b$, except for up to $\frac{k}{5}$ of them, which is smaller than a third of the size of both A and B.

Let q^t be the probability to reach x from i^* using exactly t moves of the pebble. For $t \ge i+1$ we have that

$$q^{t} = p^{t} - p^{t-1} = \frac{2F_{t+1}^{(i)} - F_{t+2}^{(i)}}{2^{t}} = \frac{F_{t+1-i}^{(i)}}{2^{t}} = 2^{-i-1}(1 - p^{t-1-i}).$$

(To have a sequence of i tails after precisely t coin flips for t > i, we need to have failed to get that many tails in a row for the first t - 1 - i coin flips and then gotten a head followed by i tails, which is also what our expression tells us.)

We see that q^t is decreasing for $t \ge i + 1$, because p^t is increasing. We can therefore calculate the probability to end up at x using a specific amount of time compared to all other times in A as

$$\frac{q^{\frac{k}{5}}}{q^{\frac{4k}{5}}} = \frac{2^{-i-1}(1-p^{\frac{k}{5}-1-i})}{2^{-i-1}(1-p^{\frac{4k}{5}-1-i})}$$

$$= \frac{F_{\frac{k}{5}+1-i}^{(i)}}{\frac{2^{\frac{k}{5}+1-i}}{2^{\frac{4k}{5}-1-i}}}$$

$$\geq \frac{F_{\frac{k}{5}+1-i}^{(i)}}{(2-2^{-i-1})^{\frac{3k}{5}}F_{\frac{k}{5}+1-i}^{(i)}}$$

$$= (1-2^{-i-2})^{-\frac{3k}{5}}$$

$$= ((1-2^{-i-2})^k)^{-\frac{3}{5}}$$

$$\geq e^{\frac{3}{40}},$$

where we used Lemma 3 for the first inequality and Lemma 5 for the second.

We can show similarly that all q^t for t being in B are also equal up to a factor of $e^{\frac{3}{40}}$. Hence, the probability to reach x from i^* with t time remaining for $t-1 \in A$ is nearly uniformly distributed over A (up to a factor of $e^{\frac{3}{40}}$). Similar with t-1 in B. Therefore we can pick an ϵ_1 (independent of i) such that $\sigma_a(m^t, x) = (x, h)$ for all but $\frac{1}{10}$ of the t's in A. Similar, we can pick an ϵ_2 (independent of i) such that $\sigma_a(m^t, x) = (x, i)$ for all but $\frac{1}{10}$ of the t's in B.

of i) such that $\sigma_a(m^t, x) = (x, i)$ for all but $\frac{1}{10}$ of the t's in B. By using $\epsilon = \min(\epsilon_1, \epsilon_2)$ both $\frac{9}{10}$ of all t in A have that $\sigma_a(m^t, x) = (x, h)$ and $\frac{9}{10}$ of all t in B have that $\sigma_a(m^t, x) = (x, i)$. But this contradicts that we had a one to one map that mapped at least two thirds of all m^a for a in A to some memory m^b for b in B such that $m^a = m^b$ (and at least two thirds of the b's got mapped to).

Hence all ϵ -optimal counter-based strategies uses memory at least $\frac{k}{5}$. The result then follows from $k \geq 2^{i-2} + i$ from Lemma 4.

Theorem 10 (Lower bound) For all sufficiently small $\epsilon > 0$ and all $n \ge 5$, there is a FMDP with n states, where all ϵ -optimal counter-based strategies require memory size at least $\Omega(\log \log \epsilon^{-1} + n)$.

Proof The proof is a simple combination of the two lower bounds in Lemma 2 and Lemma 9. \square

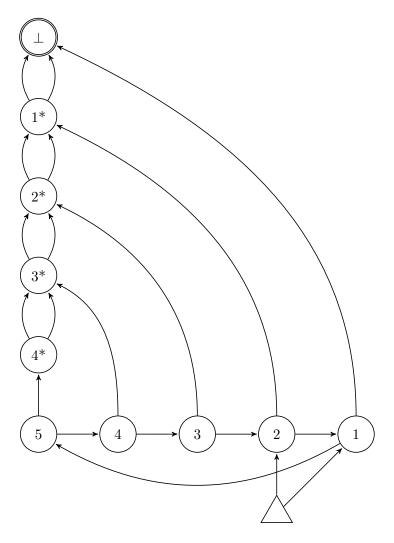


Figure 3: The MDP G_5 . Circle vertices are the coin toss states. The triangle vertex is the Max state. The vertex \perp is the terminal state.

4 A lower bound on the period of optimal strategies in MDPs

We will in this section show that there exist FMDPs G, with n states, such that all optimal strategies can be implemented using a counter-based strategy, and the period is greater than $2^{\Omega(\sqrt{n \log n})}$. We will create such FMDPs in two steps. First we will construct a family, such that the i'th member requires that one state uses one action every $\Theta(i)$ steps and in all other steps uses the other action. There is an illustration of a member of that family in Figure 3. Afterwards we will play many such games in parallel, which will ensure that a large period is needed for all optimal strategies. There is an illustration of such a game in Figure 4.

Let G_p , $p \in \{2, 3, ...\}$ be the following FMDP, with 2p - 1 coin toss states and one max state. The coin toss states are divided into the sets $\{1^*, 2^*, ..., (p-1)^*\}$ and $\{1, 2, ..., p\}$. To simplify the following description let state 0^* denote the \bot terminal state. A description of G is then

- State i^* has state $(i-1)^*$ as both its successors.
- State i has state $(i-1)^*$ and (i-1) as successors, except state 1 which has \perp and state p as successors.
- The max state has 1 and 2 as successors.

There is an illustration of G_5 in Figure 3.

Lemma 11 Let $p \ge 2$ be given. State i has value $1 - 2^{-f_i(k)}$ in G_p^k for k > 0, where $f_i(k)$ is the function $f_i(k) = \max_{k' \le k \land k' \mod p = i}(k', 0)$.

Proof It is easily seen by induction that i^* has value 1 in G_p^i . Note that $f_i(k) = i$ for $k \mod p = i$. The proof will be by induction in k. There will be one base case and two induction cases, one for $1 < k \le p$ and one for k > p. It is easy to see that state 1 has value $\frac{1}{2} = 1 - \frac{1}{2} = 1 - 2^{-f_1(1)}$ in G_p^1 and state j for $j \ne 1$ has value 0. That settles the base case.

For $1 < k \le p$. Neither of the successors of state j, for $j \ne k$, has changed values from G_p^{k-2} to G_p^{k-1} . For state k, both its successors has changed value. The value of state $k-1^*$ has become $\operatorname{val}(G_p^{k-1})_{k-1^*} = 1$ and the value of state k-1 has become $\operatorname{val}(G_p^{k-1})_{k-1} = 1 - 2^{-f_{k-1}(k-1)}$. The value of state k is then

$$\operatorname{val}(G_p^k)_k = \frac{1+1-2^{-f_{k-1}(k-1)}}{2} = \frac{1+1-2^{-(k-1)}}{2} = 1-2^{-(k-1)-1} = 1-2^{-f_k(k)}.$$

For p < k. Let i be $k \mod p$. Neither of the successors of state j, for $j \neq i$, has changed values from G_p^{k-2} to G_p^{k-1} . The value of state $i' = i - 1 \mod p$, in iteration k - 1 is $\operatorname{val}(G_p^{k-1})_{i'} = 1 - 2^{-f_{i'}(k-1)}$. The value of state i is then

$$\operatorname{val}(G_p^k)_i = \frac{1 + 1 - 2^{-f_{i'}(k-1)}}{2} = \frac{1 + 1 - 2^{-(k-1)}}{2} = 1 - 2^{-(k-1)-1} = 1 - 2^{-f_i(k)}.$$

The desired result follows.

The idea behind the construction of F_k is that to find the state of the largest value among 1 and 2, in G_p^T , for $p \geq 2$ and $T \geq 1$, we need to know if $T \mod p = 1$ or not. Let p_i be the *i*'th smallest prime number. The FMDP F_k is as follows: F_k consists of a copy of G_{p_i} for $i \in \{1, \ldots, k\}$. Let the max state in that copy of G_{p_i} be m_i . There is a illustration of F_2 in Figure 4.

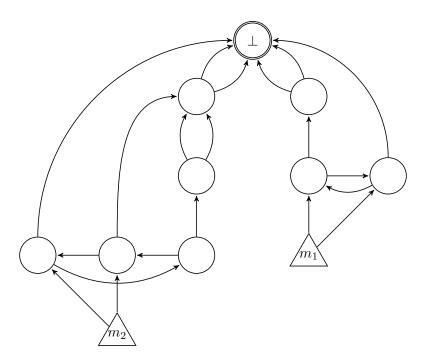


Figure 4: The FMDP F_2 . Circle vertices are the coin toss states. Triangle vertices are the max states. The vertex \perp is the terminal state.

We will now show that all optimal strategies for F_k are subsets of counter-based strategies with a period defined by k. Afterwards we will show that the number of states in F_k can also be expressed in terms of k. At the end we will use those two lemmas to get to our result.

Lemma 12 Any optimal strategy $\sigma(k, T')$ in F_k is an finite memory counter-based strategies with period $P = \prod_{i \in \{1,...,k\}} p_i$, where p_i is the i'th smallest prime number.

Proof Let i be some number in $\{1, \ldots, k\}$. The lone optimal choice for m_i and T' > 0 is to use the action that goes to state 1 in G_{p_i} if T mod $p_i = 1$ and otherwise to use the action that goes to state 2 in G_{p_i} by Lemma 11. Hence, by the Chinese remainder theorem there are precisely P steps between each time any optimal strategy uses the action that goes to 1 in all m_i 's. That is, any optimal strategy must do the same action at least every P steps. Furthermore it is also easy to see that any optimal strategy must do the same at most every P steps, by noting that T + P mod p_i is 1 if and only if T mod p_i is 1 and again applying Lemma 11. A strategy that does the same every P steps can be expressed by a counter-based strategy with period P, which also uses memory at most P.

Lemma 13 The number of states in F_k is $2\sum_{i\in\{1,\ldots,k\}} p_i$.

Proof For any i, G_{p_i} consists of $2p_i$ states. F_k therefore consists of $2\sum_{i\in\{1,\ldots,k\}}p_i$ states.

Theorem 14 There are FMDPs G, with n states, where all optimal strategies are finite memory counter-based strategies with period $2^{\Omega(\sqrt{n \log n})}$.

Proof Let n be such that there exists a game F_k with n states. Note that for any number there is always a larger number, a, such that F_b has a states for some b.

By Lemma 13, we have that $n = 2\sum_{i \in \{1,...,k\}} p_i$. By the prime number theorem (see e.g. Newman [6]) we have that $\sum_{i \in \{1,...,k\}} p_i = \sum_{i \in \{1,...,k\}} o(k \log k) = o(k^2 \log k)$.

Let $f(x) = x^2 \log x$ for x > 1. The function f(x) is strictly monotone increasing and hence, has an inverse function. Let that function be $f^{-1}(y)$. We have that $f^{-1}(y) \ge \sqrt{\frac{y}{\log y}}$, for $y \ge 2$, because

$$f^{-1}(y) \ge \sqrt{\frac{y}{\log y}} \quad \Leftarrow \quad f(f^{-1}(y)) \ge f(\sqrt{\frac{y}{\log y}})$$

$$\Leftarrow \quad y \ge (\sqrt{\frac{y}{\log y}})^2 \log(\sqrt{\frac{y}{\log y}})$$

$$\Leftarrow \quad y \ge \frac{y}{\log y} \log(\sqrt{\frac{y}{\log y}})$$

$$\Leftarrow \quad y \ge \frac{y}{\log y} \log y$$

$$\Leftarrow \quad y \ge y$$

Here, the first \Leftarrow follows by taking f^{-1} on both sides. The function f^{-1} is strictly monotone increasing, because f(x) was. The fourth \Leftarrow follows from $y \ge \sqrt{\frac{y}{\log y}}$ for $y \ge 2$ and log being monotone increasing.

Therefore, let $g(k) = 2 \sum_{i \in \{1,...,k\}} p_i$, then $g^{-1}(n) = \Omega(\sqrt{\frac{n}{\log n}})$. By Lemma 12, we have that the period is $\prod_{i \in \{1,...,k\}} p_i$. Trivially we have that

$$\prod_{i \in \{1, \dots, k\}} p_i \ge \prod_{i \in \{1, \dots, k\}} i = k! = 2^{\Omega(k \log k)}$$

We now insert $\Omega(\sqrt{\frac{n}{\log n}})$ in place of k and get

$$\prod_{i \in \{1,\dots,k\}} p_i = 2^{\Omega(\Omega(\sqrt{\frac{n}{\log n}})\log(\Omega(\sqrt{\frac{n}{\log n}})))} = 2^{\Omega(\sqrt{\frac{n}{\log n}}(\log n - \log\log n))} = 2^{\Omega(\sqrt{n\log n})}$$

The result follows.

5 Conclusion

In the present paper we have considered properties of finite-horizon Markov decision processes and simple stochastic games. The ϵ -optimal strategies considered in Section 3 indicates the hardness of playing such games with a short horizon. The concept of period from Section 4 indicates the hardness of playing such games with a long horizon. Along with our lower bound from Section 4 we conjecture the following:

Conjecture 15 All FSSGs have an optimal strategy, which is an finite memory counter-based strategy, with period at most 2^n .

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